

Spectrum of Periodically Correlated Fields

Dominique Dehay

Institut de Recherche Mathématique de Rennes, CNRS umr 6625,

Université Rennes 2, cs 24307, 35043 Rennes, FRANCE; dominique.dehay@univ-rennes2.fr

Harry Hurd

Department of Statistics, University of North Carolina,

Chapel Hill, NC 27599-2630, USA; hurd@stat.unc.edu

Andrzej Makagon *

Department of Mathematics, Hampton University

Hampton, VA 26668, USA; andrzej.makagon@hamptonu.edu

July 9, 2013

Abstract

The paper deals with Hilbert space valued fields over any locally compact Abelian group G , in particular over $G = \mathbb{Z}^n \times \mathbb{R}^m$, which are periodically correlated (PC) with respect to a closed subgroup of G . PC fields can be regarded as multi-parameter extensions of PC processes. We study structure, covariance function, and an analogue of the spectrum for such fields. As an example a weakly PC field over \mathbb{Z}^2 is thoroughly examined.

1 Introduction

Periodically correlated (PC) processes and sequences have been studied for almost half of the century and at present they are very well understood mainly due to works of Gladyshev [12, 13], Hurd [17, 18, 19, 20, 22] and other authors [5, 16, 26, 27, 28, 29, 30]. A summary of the theory of PC sequences can be found in [23]. Surprisingly, there are only several works [2, 3, 4, 6, 7, 10, 11, 21, 36] dealing with PC fields, and each one concentrates on a particular type, namely coordinate-wise strong periodicity. An intention of this paper is to sketch a unified theory of fields over any locally compact Abelian (LCA) group G which are periodically correlated with respect to an arbitrary closed subgroup K of G . We emphasize the case of $G = \mathbb{Z}^n \times \mathbb{R}^m$ to illustrate the results. This work

*The paper was partially written during the author's stay at Université Rennes 2 - Haute Bretagne, Rennes, France, in June 2011.

¹2010 Mathematics Subject Classifications : 60G12

²Key Words : Periodically correlated processes, Stochastic processes and fields, Harmonizable processes, Spectrum, Shift operator, LCA group, Fourier transform.

includes stationary fields as well the weakly periodically correlated fields, that is the fields whose covariance function exhibits periodicity (or stationarity) in fewer directions than the dimension of the group. In the latter case we assume a certain integrability condition (see Definition 3) in order to develop some simple spectral analysis of those fields. A work in progress treats the case where this condition is not satisfied.

The paper is organized as follows. In the remaining part of this section we introduce notation and vocabulary used in the paper, review needed facts from harmonic analysis on LCA groups, and outline the theory of one-parameter PC processes. In the next three sections we study the covariance function, the notion of the spectrum, and the structure of a K -periodically correlated field. These sections include the main results of the paper (Theorems 1, 2 and 5). The last section contains examples that illustrate the theory developed. In particular Example 2 gives a complete analysis of the weakly periodically correlated fields over \mathbb{Z}^2 , introduced in [21].

Background

To avoid confusion and to set the notations of the paper we recall some features of group theory, Haar measures, Fourier transform, and periodic functions. For more information on these subjects the authors refer to [14, 33, 35].

1. Quotient groups, cross-sections, Haar measure, and Fourier transform. Let G be an additive locally compact Abelian (LCA) group, \widehat{G} be its dual (group of continuous characters), and let $\langle \chi, t \rangle$ denote the value of a character $\chi \in \widehat{G}$ at $t \in G$. The dual \widehat{G} can be given a topology that makes it an LCA group such that $\widehat{(\widehat{G})} = G$. Let K be a closed subgroup of G . The symbol G/K will stand for the quotient group and $\widehat{(G/K)}$ for its dual. Let ι denote the natural homomorphism of G onto G/K , $\iota(t) := t + K$, and ι^* be its dual map $\iota^* : \widehat{G/K} \rightarrow \widehat{G}$, defined as $\langle \iota^*(\eta), t \rangle = \langle \eta, (t + K) \rangle$ for $\eta \in \widehat{G/K}$ and $t \in G$. The mapping ι^* is injective and continuous, and for each $\eta \in \widehat{G/K}$, $\langle \iota^*(\eta), \cdot \rangle$ is a K -periodic function on G (see below). Consequently $\widehat{G/K}$ can be identified with a closed subgroup Λ_K of \widehat{G} consisting of the elements $\lambda \in \widehat{G}$ such that $\langle \lambda, t \rangle = 1$ for any $t \in K$. In the sequel we use the notation $\langle \lambda, t \rangle := \langle \lambda, \iota(t) \rangle$, for all $\lambda \in \Lambda_K$ and $t \in G$. By $\mathcal{B}(G)$ we denote the σ -algebra of Borel sets on G . A *cross-section* ξ for G/K is a mapping $\xi : G/K \rightarrow G$ such that

- (i) ξ is Borel,
- (ii) $\xi(G/K)$ is a measurable subset of G ,
- (iii) $\xi(0) = 0$ and $\xi \circ \iota(t) \in t + K$ for all $t \in G$, where $t + K := \{t + k : k \in K\}$.

For existence and other properties of a cross-section please see [24, 37]. For each cross-section ξ for G/K , the sets $k + \xi(G/K)$, $k \in K$, are disjoint and their union is G , and hence each element $t \in G$ has a unique representation $t = k(t) + \xi(\iota(t))$, where $k(t) \in K$. Note that the function ξ is not additive, that is $\xi(x + y)$ may be different than $\xi(x) + \xi(y)$, $x, y \in G/K$.

Any LCA group has a nonnegative translation-invariant measure, unique up to a multiplicative constant, called a Haar measure. The Haar measures on G and \widehat{G} can be normalized in such a way that the following implication holds

$$\begin{aligned} \text{if } f \in L^1(G), \quad \widehat{f}(\chi) &:= \int_G \langle \chi, t \rangle f(t) \hbar_G(dt) \quad \text{for } \chi \in \widehat{G}, \quad \text{and} \quad \widehat{f} \in L^1(\widehat{G}) \\ \text{then } f(t) &= \int_{\widehat{G}} \overline{\langle \chi, t \rangle} \widehat{f}(\chi) \hbar_{\widehat{G}}(d\chi) \quad \text{for a.e. } t \in G. \end{aligned}$$

The function \widehat{f} above is called the *Fourier transform* of f . Here and what follows $L^1(G)$ stands for the space of complex functions on G which are integrable with respect to \hbar_G , and \hbar_G denotes the normalized Haar measure on the group indicated in the subscript. Note that the normalization of the Haar measures of G and \widehat{G} is not unique. We follow the usual convention that if G is compact and infinite then the normalization is such that $\hbar_G(G) = 1$; if G is discrete and infinite then the normalized Haar measure of any single point is 1; if G is both compact and finite then its dual is also and the Haar measure on G is normalized to have a mass 1 while the Haar measure on \widehat{G} is counting measure. The normalized Haar measure on \mathbb{R} is the Lebesgue measure divided by $\sqrt{2\pi}$. Finally, if K is a closed subgroup of G then the normalized Haar measures satisfy Weil's formula

$$\int_{G/K} \left(\int_K f(k+s) \hbar_K(dk) \right) \hbar_{G/K}(ds) = \int_G f(t) \hbar_G(dt), \quad f \in L^1(G). \quad (1)$$

The inner integral above depends only on the coset $\dot{s} := s + K$. See e.g. [33, Section III.3.3].

If $f \in L^1(G)$ then \widehat{f} is a continuous bounded function on \widehat{G} but not necessarily integrable. The Fourier transform, which is customarily denoted by the integral $\widehat{f}(\chi) = \int_G \langle \chi, t \rangle f(t) \hbar_G(dt)$ (even if f is not integrable) extends from $L^1(G) \cap L^2(G)$ to an isometry from $L^2(G)$ onto $L^2(\widehat{G})$ (Plancherel theorem [35]). If there is a danger of confusion we will recognize the difference by writing

$$\widehat{f}(\chi) \stackrel{L^2}{=} \int_G \langle \chi, t \rangle f(t) \hbar_G(dt), \quad f \in L^2(G).$$

In the sequel we say that *the inverse formula holds for f* if the function f is the inverse Fourier transform of \widehat{f} . If both f and \widehat{f} are integrable then clearly the inverse formula holds for both. Also if G is discrete and $f \in L^2(G)$, then the inverse formula holds for f . Indeed, in this case $\widehat{f} \in L^1(\widehat{G})$ because \widehat{G} is compact, and hence $f(t) = \int_{\widehat{G}} \overline{\langle \chi, t \rangle} \widehat{f}(\chi) \hbar_{\widehat{G}}(d\chi)$ for all $t \in G$.

For a separable Hilbert space \mathcal{H} with inner product $(\cdot, \cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$, let $L^p(G; \mathcal{H}) := L^p(G, \hbar_G; \mathcal{H})$, $p = 1$ or 2 , be the space of \mathcal{H} -valued fields on G which are p -integrable with respect to Haar measure \hbar_G , that is, $f \in L^p(G; \mathcal{H})$ means that $f : G \rightarrow \mathcal{H}$ is \hbar_G -measurable and the real-valued function $t \mapsto \|f(t)\|_{\mathcal{H}}^p$ is integrable with respect to \hbar_G . It is well known that the space $L^1(G; \mathcal{H})$ is a Banach space with the norm $\|f\|_{L^1} := \int_G \|f(t)\|_{\mathcal{H}} \hbar_G(dt)$, $f \in L^1(G; \mathcal{H})$, and the space $L^2(G; \mathcal{H})$ is a separable Hilbert space with the inner product $(f, g)_{L^2} := \int_G (f(t), g(t))_{\mathcal{H}} \hbar_G(dt)$, $f, g \in L^2(G; \mathcal{H})$. See e.g. [9, Chapter III] (see also [8, 15, 34]). Whenever $f \in L^1(G; \mathcal{H})$ then f is Bochner integrable (also called strongly integrable) and its Fourier transform exists. Furthermore Plancherel theorem applies and defines an isometry from $L^2(G; \mathcal{H})$ onto $L^2(\widehat{G}; \mathcal{H})$ (one-to-one), so $\widehat{f} \in L^2(\widehat{G}; \mathcal{H})$ is also well defined for $f \in L^2(G; \mathcal{H})$.

2. Periodic functions. Given G and a closed subgroup K of G , it is natural to call a function f defined on G to be K -periodic if

$$f(t+k) = f(t) \quad \text{for all } t \in G \text{ and } k \in K.$$

In this case, the function f is constant on cosets of K . Hence a function f on G is K -periodic if and only if f is of the form $f = f_K \circ \iota$, where f_K is a function on G/K . The concrete realization $\Lambda_K := \iota^*(\widehat{G/K}) \subset \widehat{G}$ of $\widehat{G/K}$ as a subgroup of \widehat{G} will be in the sequel called *the domain of the spectrum of f* . Note that Λ_K is not determined uniquely by f , for a K -periodic function can be at the same time periodic with respect to a larger subgroup $K' \supset K$; in other words we will not be assuming that K is the "smallest" period of f .

If $f \in L^1(G/K; \mathcal{H})$, \mathcal{H} being the set of complex numbers \mathbb{C} or any separable Hilbert space, we consider the Fourier transform of f_K at $\lambda \in \Lambda_K$

$$\widehat{f_K}(\lambda) := \int_{G/K} \langle \lambda, x \rangle f_K(x) \hbar_{G/K}(dx) \quad (2)$$

that will be referred to as the *spectral coefficient of f at frequency $\lambda \in \Lambda_K$* .

A couple of remarks regarding the above definition and its relation to the standard notions of the spectrum and its domain are certainly due here. The word *spectrum* comes originally from physics, operator theory, and more recently from signal processing. It is widely used in the theory of second order stochastic processes. Intuitively, the spectrum of a scalar function f is a Fourier transform of f in whatever sense it exists. If f is a locally integrable function on $G = \mathbb{Z}^n \times \mathbb{R}^m$ then the spectrum F of f is a Schwartz distribution on \widehat{G} , which is a functional on a certain space of functions on \widehat{G} determined by the relation $F(\widehat{\phi}) = \int_G \phi(t) f(t) \hbar_G(dt)$, where ϕ runs over the set of compactly supported functions on G which are infinitely many times differentiable in last m variables. One can show that if f is additionally K -periodic, then the support of F (as defined in [34]) is a subset of Λ_K . This rationalizes the name "*domain of the spectrum*" that we have assigned for Λ_K , as well the phrase "*the spectrum sits on Λ_K* " which we will use sometimes. The first task in understanding the spectrum of a K -PC field is thus to identify the domain of its spectrum or its second order spectrum. (See below).

The coefficient $\widehat{f_K}(\lambda)$ defined in (2) represents an "amplitude" of the harmonic $\langle \lambda, \cdot \rangle$ in a spectral decomposition of f . Indeed, if f_K and $\widehat{f_K}$ are integrable, then $f_K(x) = \int_{\Lambda_K} \overline{\langle \lambda, x \rangle} \widehat{f_K}(\lambda) \hbar_{\Lambda_K}(d\lambda)$, $x \in G/K$, and as a consequence of Weil's formula (1) and the fact that $\langle \lambda, t \rangle = \langle \lambda, \iota(t) \rangle$, $t \in G$, $\lambda \in \Lambda_K$, we conclude that

$$f(t) = \int_{\Lambda_K} \overline{\langle \lambda, t \rangle} \widehat{f_K}(\lambda) \hbar_{\Lambda_K}(d\lambda), \quad t \in G. \quad (3)$$

If $\widehat{f_K}$ is not integrable, then equality (3) holds only for \hbar_G -almost every $t \in G$ or is not valid as stated, but a_λ still retains its interpretation.

For illustration suppose that f is a continuous scalar function on \mathbb{R} which is periodic with period $T > 0$, that is such that $f(t) = f(t+T)$ for every $t \in \mathbb{R}$. In this case $G = \mathbb{R}$, $K = \{kT : k \in \mathbb{Z}\}$, the

quotient group G/K can be identified with $[0, T)$ with addition *modulo* T , the mapping ι is defined as $\iota(t) = [t]_T$, the remainder in integer division of t by T , and the identity $\xi(x) = x$, $x \in [0, T)$, is the most natural cross-section for G/K . The function f_K is defined as $f_K(x) = f(\xi(x)) = f(x)$, $x \in [0, T)$. The dual of G/K is identified with the subgroup $\Lambda_K = \{2\pi j/T : j \in \mathbb{Z}\}$ of \mathbb{R} , and with this identification $\langle \lambda, \iota(t) \rangle = \langle \lambda, t \rangle = e^{-i\lambda t}$, $\lambda \in \Lambda_K$, $t \in \mathbb{R}$. The normalized Haar measures on $[0, T)$ and Λ_K are the Lebesgue measure divided by T and the counting measure, respectively. The domain of the spectrum of f is therefore the set Λ_K . The spectral coefficient \hat{f}_j of f at $\lambda = 2\pi j/T$, is given by

$$\hat{f}_j = \frac{1}{T} \int_0^T e^{-i2\pi jt/T} f(t) dt, \quad j \in \mathbb{Z}.$$

Note that the sequence $\{\hat{f}_j\}$ is square-summable and consequently $f(t) = \sum_{j=-\infty}^{\infty} e^{i2\pi jt/T} \hat{f}_j$, where the series above converges in $L^2[0, T]$, so in $L^2([-A, A])$ for every $0 < A < \infty$. The spectrum F of f is defined by the relation $F(\hat{\phi}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) f(t) dt$. If ϕ is an infinitely times differentiable with compact support then

$$F(\hat{\phi}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) f(t) dt = \int_{\mathbb{R}} \hat{\phi}(t) F(dt),$$

where $F = \sum_{j=-\infty}^{\infty} \hat{f}_j \delta_{\{2\pi j/T\}}$, and δ_a denotes the measure of mass 1 concentrated at $\{a\}$. The spectrum of f can be therefore identified with a σ -additive complex measure F on \mathbb{R} sitting on Λ_K and defined by $F = \sum_{j=-\infty}^{\infty} \hat{f}_j \delta_{\{2\pi j/T\}}$. If the sequence $\{\hat{f}_j\}$ is summable, then F is a finite measure, but it does not have to be in general.

2 Periodically Correlated Fields

Let \mathcal{H} be a separable Hilbert space with the inner product $(\cdot, \cdot)_{\mathcal{H}}$. In a probabilistic context the space \mathcal{H} represents the space of zero-mean complex random variables with finite variance. A (*stochastic*) *field* $X = \{X(t) : t \in G\}$ is a measurable function $X : G \rightarrow \mathcal{H}$. Let $\mathcal{H}_X := \overline{\text{span}} \{X(t) : t \in G\}$ be the smallest closed linear subspace of \mathcal{H} that contains all $X(t)$, $t \in G$. The function $\mathbf{K}_X(t, s) := (X(t), X(s))_{\mathcal{H}}$, $t, s \in G$, is referred to as the *covariance function* of the field X . A field X is called *stationary* if it is continuous and for all $t, s \in G$, the function $\mathbf{K}_X(t+u, s+u)$ does not depend on $u \in G$. If X is stationary then $\mathbf{K}_X(t+s, s) = \mathbf{K}_X(t+0, 0) =: \mathbf{R}_X(t)$, $t, s \in G$, and \mathbf{R}_X has the form

$$\mathbf{R}_X(t) = \int_{\hat{G}} \overline{\langle \chi, t \rangle} \Gamma(d\chi),$$

where Γ is a non-negative Borel measure on \hat{G} (Bochner Theorem [33, Section IV.4.4]). A field X is called *harmonizable* if there is a (complex) measure F on $\hat{G} \times \hat{G}$ such that

$$\mathbf{K}_X(t, s) = \iint_{\hat{G}^2} \overline{\langle \chi, t \rangle} \langle \beta, s \rangle F(d\chi, d\beta), \quad t, s \in G, \quad (4)$$

see [31, 32]. The measure F above is called the *second order spectral (SO-spectral) measure* of the harmonizable field X . Note that every stationary field is harmonizable with the measure F

sitting on the diagonal: $F(\Delta) = \Gamma\{\chi \in \widehat{G} : (\chi, \chi) \in \Delta\}$, $\Delta \in \mathcal{B}(\widehat{G} \times \widehat{G})$. The SO-spectrum of a harmonizable field X is the spectral measure F associated with function $\mathbf{K}_X(t, -s)$, $s, t \in G$ via relation (4). Here we adopt the terminology of "second order spectrum (SO-spectrum)" of the field X , instead of the usual term "spectrum", in order to avoid confusion with the spectrum of a periodic function (or field). By this way we point at the fact that we are considering not the field X by itself but its covariance function \mathbf{K}_X . This leads to the following definition.

Definition 1 *Let X be a continuous stochastic field over G . The second order spectrum (SO-spectrum) of the field X is the spectrum (Fourier transform, in whatever sense it may exist) of the function $G \times G \ni (t, s) \mapsto \mathbf{K}_X(t, -s)$. The domain of the SO-spectrum of the field X is defined as the domain of the spectrum of this function.*

Definition 2 *Let K be a closed subgroup of G . A field X is called K -periodically correlated (K -PC) if X is continuous and the function $G \ni u \mapsto \mathbf{K}_X(t + u, s + u)$ is K -periodic in u for all $t, s \in G$. The group K will be called the period of the PC process X .*

If $G = \mathbb{R}$ (or \mathbb{Z}) and $K = \{kT : k \in \mathbb{Z}\}$ then we will use the phrase "PC process (or PC sequence) with period $T > 0$ ", rather than K -PC field. Note that every stationary field over G is K -PC field with $K = G$. A K -PC field is labeled *strongly PC* if G/K is compact, and *weakly PC* otherwise.

For example if X is a field over \mathbb{R}^2 (or \mathbb{Z}^2) such that for every $\mathbf{s}, \mathbf{t} \in \mathbb{R}^2$ (or \mathbb{Z}^2), $\mathbf{K}_X(\mathbf{s}, \mathbf{t}) = \mathbf{K}_X(\mathbf{s} + (T_1, 0), \mathbf{t} + (T_1, 0)) = \mathbf{K}_X(\mathbf{s} + (0, T_2), \mathbf{t} + (0, T_2))$, $0 < T_1, T_2 < \infty$, then X is strongly K -PC with $K = \{(k_1 T_1, k_2 T_2) : k_1, k_2 \in \mathbb{Z}\}$. Since \mathbf{K}_X is invariant under shifts from K this leads to the existence of unitary operators U_1, U_2 in \mathcal{H}_X such that $U_1 X(\mathbf{t}) = X(t_1 + T_1, t_2)$ and $U_2 X(\mathbf{t}) = X(t_1, t_2 + T_2)$ for every $\mathbf{t} = (t_1, t_2)$. If the field X instead satisfies $\mathbf{K}_X(\mathbf{s}, \mathbf{t}) = \mathbf{K}_X(\mathbf{s} + (T_1, T_2), \mathbf{t} + (T_1, T_2))$, then X is weakly K -PC with $K = \{k(T_1, T_2) : k \in \mathbb{Z}\}$. This leads to a unitary operator U such that $UX(\mathbf{t}) = X(\mathbf{t} + (T_1, T_2))$, $\mathbf{t} = (t_1, t_2)$.

Examples of PC fields on \mathbb{Z}^2 can be constructed by a periodic amplitude or time deformation of a stationary field. Suppose $X(\mathbf{t}) = f(\mathbf{t})Y(\mathbf{t})$, $\mathbf{t} = (t_1, t_2) \in \mathbb{Z}^2$, where Y is a stationary field and f is a non-random periodic function such that $f(t_1, t_2) = f(t_1 + T_1, t_2) = f(t_1, t_2 + T_2)$. Then the field X is strongly K -PC with $K = \{(k_1 T_1, k_2 T_2) : k_1, k_2 \in \mathbb{Z}\}$. If f above instead satisfies $f(t_1, t_2) = f(t_1 + T_1, t_2 + T_2)$, then X is weakly K -PC with $K = \{k(T_1, T_2) : k \in \mathbb{Z}\}$. If the function f is two-dimensional integer valued, then the field X defined by $X(\mathbf{t}) = Y(\mathbf{t} + f(\mathbf{t}))$ will be weakly PC.

Remark that generally $\mathbf{K}_X(t + u, s + u) = \mathbf{K}_X(t - s + s + u, s + u)$, so a continuous field X is K -PC if and only if $\mathbf{K}_X(t + u, u)$ is a K -periodic function of u for every $t \in G$. If X is K -PC field then for all $t, s \in G$ there is a unique function $x \mapsto b_X(t, s; x)$ on G/K such that

$$\mathbf{K}_X(t + u, s + u) = b_X(t, s; v(u)), \quad t, s, u \in G.$$

The canonical map $\iota : G \rightarrow G/K$ is continuous and open [33, Section III.1.6], so the function $x \mapsto b_X(s, t; x)$ is continuous. Denote

$$\mathbf{B}_X(t; x) := b_X(t, 0; x), \quad t \in G, x \in G/K.$$

Note that $b_X(t, s; \iota(u)) = b_X(t - s, 0; \iota(s + u)) = \mathbf{B}_X(t - s; \iota(s + u))$ for all $t, s, u \in G$.

In this work we need the following notion to proceed to the spectral analysis.

Definition 3 *A K -PC field X over G is called G/K -square integrable if the function $\mathbf{B}_X(0; \cdot)$ is integrable with respect to the Haar measure on G/K .*

If X is a G/K -square integrable K -PC field, then from translation-invariance of the Haar measure it follows that for every $t \in G$, $b_X(t, t; \cdot)$ is $\hbar_{G/K}$ -integrable, and

$$\begin{aligned} \int_{\xi(G/K)} \|X(t + u)\|_{\mathcal{H}}^2 (\hbar_{G/K} \circ \xi^{-1})(du) &= \int_{G/K} b_X(t, t; x) \hbar_{G/K}(dx) \\ &= \int_{G/K} \mathbf{B}_X(0; x) \hbar_{G/K}(dx) = \int_{\xi(G/K)} \|X(u)\|_{\mathcal{H}}^2 (\hbar_{G/K} \circ \xi^{-1})(du) < \infty, \end{aligned}$$

where ξ is any cross-section for G/K . Also note that if $b_X(t, t; \cdot)$ is $\hbar_{G/K}$ -integrable for any $t \in G$, then by Cauchy-Schwarz inequality $b_X(t, s; \cdot)$ is $\hbar_{G/K}$ -integrable for all $t, s \in G$.

When $X = P$ is a K -periodic continuous field, then it is a PC field and we can readily prove the following equivalence

$$P \text{ is } G/K\text{-square integrable} \iff P_K \in L^2(G/K; \mathcal{H}),$$

where P_K is the field defined on G/K by $P = P_K \circ \iota$.

When X is a PC process on \mathbb{R} with period $T > 0$ (i.e. $\mathbf{K}_X(t, s) = \mathbf{K}_X(t + T, s + T)$ for all $t, s \in \mathbb{R}$), it is well known that the SO-spectrum of X can be described as a sequence of complex measures γ_j , $j \in \mathbb{Z}$, on \mathbb{R} (cf. [23]). If in addition $\sum_j \text{Var}(\gamma_j) < \infty$ then the process X is harmonizable and

$$\mathbf{K}_X(t, s) = \iint_{\mathbb{R}^2} e^{i(ut - vs)} F(du, dv), \quad (5)$$

where $F := \sum_j F_j$, and F_j is the image of γ_j via the mapping $\ell_j(u) := (u, u - 2\pi j/T)$. If $\sum_j \text{Var}(\gamma_j) = \infty$ then $F = \sum_j F_j$ can still be viewed as the SO-spectrum of X in the framework of the Schwartz distributions theory (see [28]). For more discussion about PC processes please see Example 1 in Section 6. A corresponding description of the SO-spectrum is available for PC sequences ($G = \mathbb{Z}$). Let us remark here that a PC sequence is always harmonizable, but there are continuous PC processes which are not, see e.g. [12, 13].

The above description of the SO-spectrum of a PC process, which originates from Gladyshev's papers [12, 13], can be easily extended to the case of coordinate-wise strongly periodically correlated fields over \mathbb{R}^n or \mathbb{Z}^n (see e.g. [1, 7, 6, 11, 21]). The purpose of this work is to describe the SO-spectrum of a K -periodically correlated field for any closed subgroup K of an LCA group G and as a particular case when $G = \mathbb{R}^m \times \mathbb{Z}^n$. We also briefly address the question of structure of K -PC fields.

3 Covariance Function of a PC Field

This section contains an extension of Gladyshev's description of the covariance function of one-parameter PC processes (see [12, 13]) to the case of K -PC fields. For any G/K -square integrable K -PC field X , define *the spectral covariance function of the field X* (also called cyclic covariance in signal theory, see e.g. [10]) by

$$a_\lambda(t) := \int_{G/K} \langle \lambda, x \rangle \mathbf{B}_X(t; x) \hbar_{G/K}(dx), \quad \lambda \in \Lambda_K. \quad (6)$$

Let ξ be a fixed cross-section for G/K . For each $\lambda \in \Lambda_K$ and $t \in G$ let us define an \mathcal{H}_X -valued function $Z^\lambda(t)$ on G/K by

$$Z^\lambda(t)(x) := \langle \lambda, (\iota(t) + x) \rangle X(t + \xi(x)), \quad x \in G/K. \quad (7)$$

Notice that $Z^\lambda(t)(x)$ depends on the chosen cross-section ξ . From G/K -square integrability of X it follows that for all $\lambda \in \Lambda_K$ and $t \in G$, $Z^\lambda(t)$ is an element of the Hilbert space $L^2(G/K; \mathcal{H})$.

Theorem 1 *Let X be an \mathcal{H} -valued G/K -square integrable K -PC field, and let $a_\lambda(t)$ and $Z^\lambda(t)(x)$ be as above. Then the cross-covariance function $\mathbf{K}_Z^{\lambda, \mu}(t, s) := (Z^\lambda(t), Z^\mu(s))_{\mathcal{K}}$ of the family $\{Z^\lambda(t) : \lambda \in \Lambda_K, t \in G\}$*

$$\mathbf{K}_Z^{\lambda, \mu}(t, s) = \langle \lambda, (t - s) \rangle a_{\lambda - \mu}(t - s) =: \mathbf{R}^{\lambda, \mu}(t - s). \quad (8)$$

If additionally

[A] *the function $G \ni t \mapsto a_0(t)$ is continuous at $t = 0$,*

then $\{Z^\lambda : \lambda \in \Lambda_K\}$ is a family of jointly stationary fields over G in $L^2(G/K; \mathcal{H}_X)$.

Proof. Let ξ be a fixed cross-section for G/K . Since X is G/K -square integrable and $\iota \circ \xi(x) = x$ for any $x \in G/K$, the function $B(0; \cdot)$ is $\hbar_{G/K}$ -integrable, so $Z^\lambda(t) \in \mathcal{K}$ and

$$\begin{aligned} (Z^\lambda(t), Z^\mu(s))_{\mathcal{K}} &= \int_{G/K} \langle \lambda, (\iota(t) + x) \rangle \overline{\langle \mu, (\iota(s) + x) \rangle} b_X(t, s; \iota \circ \xi(x)) \hbar_{G/K}(dx) \\ &= \int_{G/K} \langle \lambda, (\iota(t) + x) \rangle \overline{\langle \mu, (\iota(s) + x) \rangle} \mathbf{B}_X(t - s; \iota(s) + x) \hbar_{G/K}(dx) \\ &= \langle \lambda, (t - s) \rangle \int_{G/K} \langle (\lambda - \mu), y \rangle \mathbf{B}_X(t - s; y) \hbar_{G/K}(dy) \\ &= \langle \lambda, (t - s) \rangle a_{\lambda - \mu}(t - s) \end{aligned}$$

for all $s, t \in G$. In view of relation (8), in order to complete the proof it is enough to show that for every $\lambda \in \Lambda_K$, the function $G \ni t \rightarrow Z^\lambda(t) \in \mathcal{K}$ is continuous provided condition [A] is satisfied, and this is obvious since by equality (8),

$$\begin{aligned} \|Z^\lambda(t) - Z^\lambda(s)\|_{\mathcal{K}}^2 &= \mathbf{K}_Z^{\lambda, \lambda}(t, t) - \mathbf{K}_Z^{\lambda, \lambda}(t, s) - \mathbf{K}_Z^{\lambda, \lambda}(s, t) + \mathbf{K}_Z^{\lambda, \lambda}(s, s) \\ &= 2a_0(0) - \langle \lambda, (t - s) \rangle a_0(t - s) - \langle \lambda, (t - s) \rangle a_0(s - t). \blacksquare \end{aligned}$$

Proposition 1 *The condition [A] in Theorem 1 is satisfied if either*

- (i) *G is discrete, or*
- (ii) *G/K is compact, or*
- (iii) *X is bounded, and $\mathbf{B}_X(0; \cdot)^{1/2}$ is $\hbar_{G/K}$ -integrable.*

Proof. Property [A] is evident when the group G is discrete. When G/K is compact then X is clearly bounded because $\|X(t)\|_{\mathcal{H}} = \|X(\xi(x))\|_{\mathcal{H}}$ where $x = \iota(t) \in G/K$, and $x \mapsto \|X(\xi(x))\|_{\mathcal{H}}$ is continuous. Since $\hbar_{G/K}$ is finite, the continuity of the function $t \mapsto a_0(t) = \int_{G/K} \mathbf{B}_X(t; x) \hbar_{G/K}(dx)$ follows therefore from Lebesgue dominated convergence theorem.

Suppose now that $\int_{G/K} \mathbf{B}_X(0; x)^{1/2} \hbar_{G/K}(dx) < \infty$. In this case for all t, s, x

$$\begin{aligned} |\mathbf{B}_X(t; x) - \mathbf{B}_X(s; x)| &= |\mathbf{K}_X(t + \xi(x), \xi(x)) - \mathbf{K}_X(s + \xi(x), \xi(x))| \\ &\leq \|X(t + \xi(x)) - X(s + \xi(x))\|_{\mathcal{H}} \|X(\xi(x))\|_{\mathcal{H}} \leq 2 \sup_t \|X\|_{\mathcal{H}} \mathbf{B}_X(0, x)^{1/2}, \end{aligned}$$

and $\lim_{u \rightarrow t} \mathbf{B}_X(u; x) = \mathbf{B}_X(t; x)$. Hence Lebesgue dominated convergence theorem applies and we conclude that $\lim_{s \rightarrow t} a_0(s) = a_0(t)$, so condition [A] is satisfied. \blacksquare

Relation (8) in Theorem 1 can be also obtained using Gladyshev's technique, that is by showing non-negative definiteness of $[\mathbf{R}^{\lambda, \mu}(t)]_{\lambda, \mu \in \Lambda_K}$, i.e. that

$$\sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} \mathbf{R}^{\lambda_j, \lambda_k}(t_j - t_k) \geq 0, \quad (9)$$

for any finite set of complex numbers $\{c_1, \dots, c_n\}$. Our method, which is an adaptation of the technique used in [27], has the advantage that it gives an explicit construction of an associated stationary family of fields. We want to point out here that even in the case of PC processes on \mathbb{R} with period T not every matrix function $[\mathbf{R}^{m, n}(t)]_{m, n \in \mathbb{Z}}$ with continuous entries, which is non-negative definite in the sense of (9) and such that $\mathbf{R}^{m, n}(t) e^{i2\pi mt/T}$ depends only on $m - n$, is associated with a continuous PC process of period T through the relation (8). To achieve the one-to-one correspondence one has to consider not necessarily continuous PC processes (see e.g. [27]).

To complete the analysis of the family of fields $\{Z^\lambda : \lambda \in \Lambda_K\}$ defined by (7), consider the space $\mathcal{H}_Z := \overline{\text{span}} \{Z^\lambda(t) : \lambda \in \Lambda_K, t \in G\}$. Clearly \mathcal{H}_Z is a subspace of $L^2(G/K; \mathcal{H}_X)$. In the case where $G = \mathbb{Z}^n \times \mathbb{R}^m$, these Hilbert spaces coincide. More precisely

Proposition 2 *Let X be an \mathcal{H} -valued G/K -square integrable K -PC field, and Z^λ be defined as above by (7). Assume that the LCA group G admits a countable dense subset, which is true when $G = \mathbb{Z}^n \times \mathbb{R}^m$. Then $\mathcal{H}_Z = L^2(G/K; \mathcal{H}_X)$.*

Proof. We know that $\mathcal{H}_Z \subset L^2(G/K; \mathcal{H}_X)$. To show the equality, let $f \in L^2(G/K; \mathcal{H}_X)$ be such that $(Z^\lambda(t), f)_K = 0$ for all $\lambda \in \Lambda_K$ and $t \in G$, i.e.

$$\int_{G/K} \langle \lambda, (\iota(t) + x) \rangle (X(t + \xi(x)), f(x))_{\mathcal{H}} \hbar_{G/K}(dx) = 0, \quad \lambda \in \Lambda_K, t \in G.$$

Since $\Lambda_K \sim \widehat{G/K}$ and $\langle \lambda, \iota(t) \rangle \neq 0$, the scalar product $(X(t + \xi(x)), f(x))_{\mathcal{H}} = 0$, for $\hbar_{G/K}$ -almost every $x \in G/H$ and for every $t \in G$ [14, Theorem 23.11]. This implies that for each t there is a negligible Borel subset Ξ_t of G/K such that $X(t + \xi(x)) \perp_{\mathcal{H}} f(x)$ for every $x \notin \Xi_t$. Note that for every x , $\overline{\text{span}}\{X(t + \xi(x)) : t \in G\} = \overline{\text{span}}\{X(t) : t \in G\} = \mathcal{H}_X$. If G admits a countable dense subset G^* , which is true in the case when $G = \mathbb{Z}^m \times \mathbb{R}^n$, then from continuity of X , it follows that also $\overline{\text{span}}\{X(t + \xi(x)) : t \in G^*\} = \mathcal{H}_X$. Therefore $f(x) \perp_{\mathcal{H}} \mathcal{H}_X$ for all x which are not in the negligible set $\bigcup_{t \in G^*} \Xi_t$. Hence $f(x) = 0$ for $\hbar_{G/K}$ -almost every $x \in G/K$ and Proposition 2 is proved. \blacksquare

4 SO-spectrum of a PC Field

Let X be a G/K -square integrable K -PC over G and let $\mathbf{K}_X(t, s)$ be its covariance function. The objective is to describe the domain of the SO-spectrum of X , which by definition (see Definition 1) is the domain of the spectrum of the function $\Phi(t, s) = \mathbf{K}_X(t, -s)$.

First we give a description of the domain of the domain of the SO-spectrum of the PC field X in the simplest case where G/K is compact.

Lemma 1 *Let X be a G/K -square integrable K -PC over G , and let $\Lambda_K = \iota^*(\widehat{G/K}) \subseteq \widehat{G}$. Then the domain of the SO-spectrum of X is the subgroup L of $\widehat{G} \times \widehat{G}$ given by*

$$L = \{(\gamma, \gamma - \lambda) : \lambda \in \Lambda_K, \gamma \in \widehat{G}\}.$$

Note that L can be viewed as the union of *hyperplanes*

$$L = \bigcup_{\lambda \in \Lambda_K} L_\lambda \quad \text{where} \quad L_\lambda := \{(\gamma, \gamma - \lambda) : \gamma \in \widehat{G}\}. \quad (10)$$

Proof. Since $\mathbf{K}_X(t + u, s + u)$ is K -periodic in u , the function $\Phi(t, s) = \mathbf{K}_X(t, -s)$ is itself a periodic function on $G \times G$ with the period $D = \{(k, -k) : k \in K\} \subseteq G \times G$. The domain of the SO-spectrum of X is therefore the dual group of $(G \times G)/D$ viewed as a subgroup of $\widehat{G} \times \widehat{G}$. Note that the subgroup D is the image of the subgroup $\{0\} \times K$ through the isomorphism $\mathcal{D} : G \times G \ni (t, s) \mapsto (t + s, -s) \in G \times G$, and this induces an isomorphism from the quotient group $(G \times G)/D$ onto $(G \times G)/(\{0\} \times K)$. Furthermore, since $(G \times G)/(\{0\} \times K) = G \times G/K$ and its dual is $\widehat{G} \times \Lambda_K$, we deduce that the dual of $(G \times G)/D$ can be identified with the subgroup L of $\widehat{G} \times \widehat{G}$ consisting of the elements of the form $(\chi, \chi - \lambda)$, $\chi \in \widehat{G}$, $\lambda \in \Lambda_K$. \blacksquare

We have not used the fact that \mathbf{K}_X is a covariance function of a process. It turns out that this additional property of \mathbf{K}_X (i.e. the fact that it is nonnegative definite) implies that the "part of the SO-spectrum" that sits on each L_λ is a measure. We want to point out that the set Λ_K may be uncountable.

Theorem 2 Suppose that X is a G/K -square integrable K -PC field that satisfies the condition [A] of Theorem 1. Then for every $\lambda \in \Lambda_K$ there is a unique Borel complex measure γ_λ on \widehat{G} such that

$$a_\lambda(t) = \int_{\widehat{G}} \overline{\langle \chi, t \rangle} \gamma_\lambda(d\chi), \quad t \in G. \quad (11)$$

Furthermore $\sup_\lambda \text{Var}(\gamma_\lambda) < \infty$ and for each $\lambda \in \Lambda_K$, γ_λ is absolutely continuous with respect to γ_0 .

In this paper by a *representation of G in a Hilbert space \mathcal{K}* we mean a weakly continuous group $\mathcal{U} := \{U^t : t \in G\}$ of unitary operators in \mathcal{K} (see [14, Section 22]). In this case there exists a weakly countably additive orthogonally scattered (w.c.a.o.s) Borel operator-valued measure E on \widehat{G} such that for every Borel set Δ the operator $E(\Delta)$ is an orthogonal projection in \mathcal{K} , and for every $u, v \in \mathcal{K}$,

$$(U^t u, v)_\mathcal{K} = \int_{\widehat{G}} \overline{\langle \chi, t \rangle} (E(d\chi)u, v)_\mathcal{K}, \quad t \in G.$$

"Orthogonally scattered" means that $(E(\Delta_1)u, E(\Delta_2)v)_\mathcal{K} = 0$ for all disjoint Δ_1, Δ_2 and $u, v \in \mathcal{K}$. The measure E will be referred to as the *spectral resolution of the unitary operator group \mathcal{U}* .

Proof of Theorem 2. The joint stationarity of the fields $\{Z^\lambda : \lambda \in \Lambda_K\}$ defined by (7), implies that each $Z^\lambda(t) = U^t Z^\lambda(0)$ where $\mathcal{U} := \{U^t : t \in G\}$ is the common shift operators group. Condition [A] guarantees the continuity of the representation \mathcal{U} of G in $L^2(G/K; \mathcal{H}_X)$, and hence

$$Z^\lambda(t) = \int_{\widehat{G}} \overline{\langle \chi, t \rangle} E(d\chi) Z^\lambda(0),$$

where E is the spectral resolution of \mathcal{U} . Therefore for every $\lambda, \mu \in \Lambda_K$ there is a complex measure $\Gamma^{\lambda, \mu}$ on \widehat{G} such that $\mathbf{R}_Z^{\lambda, \mu}(t) = \int_{\widehat{G}} \overline{\langle \chi, t \rangle} \Gamma^{\lambda, \mu}(d\chi)$, namely, $\Gamma^{\lambda, \mu}(\Delta) = (E(\Delta)Z^\lambda(0), Z^\mu(0))_\mathcal{K}$, where $\mathcal{K} := L^2(G/K; \mathcal{H}_X)$. Consequently, from relations (6) and (8) we conclude that

$$a_\lambda(t) = \int_{G/K} \langle \lambda, x \rangle \mathbf{B}_X(t; x) \hbar_{G/K}(dx) = \mathbf{R}_Z^{0, -\lambda}(t) = \int_{\widehat{G}} \overline{\langle \chi, t \rangle} \Gamma^{0, -\lambda}(d\chi).$$

Then equality (11) is satisfied with $\gamma_\lambda := \Gamma^{0, -\lambda}$. From Cauchy-Schwarz inequality we have

$$\begin{aligned} |\Gamma^{0, -\lambda}(\Delta)| &= |(E(\Delta)Z^0(0), E(\Delta)Z^{-\lambda}(0))_\mathcal{K}| \\ &\leq \sqrt{\Gamma^{0, 0}(\Delta)} \sqrt{\Gamma^{-\lambda, -\lambda}(\Delta)} = \sqrt{\Gamma^{0, 0}(\Delta)} \sqrt{\Gamma^{0, 0}(\Delta - \lambda)}, \end{aligned}$$

and we deduce the absolute continuity of γ_λ with respect to γ_0 for any $\lambda \in \Lambda_K$.

Finally note that the total variations of measures $\Gamma^{\lambda, \lambda}$, $\lambda \in \Lambda_K$, are all equal to $\Gamma^{0, 0}(\widehat{G})$. Indeed, since the measures $\Gamma^{\lambda, \lambda}$, $\lambda \in \Lambda_K$, are non-negative

$$\text{Var}(\Gamma^{\lambda, \lambda}) = \Gamma^{\lambda, \lambda}(\widehat{G}) = \mathbf{R}_Z^{\lambda, \lambda}(0) = \int_{G/K} \mathbf{B}_X(0; y) \hbar_{G/K}(dy) = \mathbf{R}_Z^{0, 0}(0) = \Gamma^{0, 0}(\widehat{G}).$$

Hence all total variations $\text{Var}(\Gamma^{\lambda, \mu}) \leq \sqrt{\Gamma^{\lambda, \lambda}(\widehat{G})} \sqrt{\Gamma^{\mu, \mu}(\widehat{G})}$, $\lambda, \mu \in \widehat{G}$, are bounded by the same constant, and in consequence all measures γ_λ , $\lambda \in \Lambda_K$, have uniformly bounded total variations. ■

Remark that when the field $X = P$ is K -periodic and G/K -square integrable, the field P_K is $\hbar_{G/K}$ -square integrable and thanks to Parseval equality, the spectral covariance function of the field P can be expressed as

$$\begin{aligned} a_\lambda^P(t) &= \int_{G/K} \langle \lambda, x \rangle (P_K(\iota(t) + x), P_K(x))_{\mathcal{H}} \hbar_{G/K}(dx) \\ &= \int_{\Lambda_K} \overline{\langle \chi, \iota(t) \rangle} (\widehat{P}_K(\chi), \widehat{P}_K(\chi - \lambda))_{\mathcal{H}} \hbar_{\Lambda_K}(d\chi) \end{aligned}$$

where \widehat{P}_K is the Fourier Plancherel transform of the field P_K . Then, we deduce that the function $\chi \mapsto (\widehat{P}_K(\chi), \widehat{P}_K(\chi - \lambda))_{\mathcal{H}}$ is the density function of the SO-spectral measure γ_λ^P of the field P with respect to \hbar_{Λ_K} ,

$$\gamma_\lambda^P(\Delta) = \int_{\Delta \cap \Lambda_K} (\widehat{P}_K(\chi), \widehat{P}_K(\chi - \lambda))_{\mathcal{H}} \hbar_{\Lambda_K}(d\chi) \quad (12)$$

for any $\Delta \in \mathcal{B}(\widehat{G})$. Particulary

$$\gamma_0^P(\Delta) = \int_{\Delta \cap \Lambda_K} \|\widehat{P}_K(\chi)\|_{\mathcal{H}}^2 \hbar_{\Lambda_K}(d\chi).$$

Notice that SO-spectral measure γ_λ^P is concentrated on $\Lambda_K \subset \widehat{G}$: $\gamma_\lambda^P(\Delta) = \gamma_\lambda^P(\Delta \cap \Lambda_K)$ for any $\Delta \in \mathcal{B}(\widehat{G})$. When in addition \widehat{P}_k is \hbar_{Λ_K} -integrable, then the fields P_K and P are harmonizable with

$$\begin{aligned} \mathbf{K}_P(t, s) &= \mathbf{K}_{P_K}(\iota(t), \iota(s)) \\ &= \iint_{\Lambda_K \times \Lambda_K} \overline{\langle \lambda, \iota(t) \rangle} \langle \mu, \iota(s) \rangle (\widehat{P}_K(\lambda), \widehat{P}_K(\mu))_{\mathcal{H}} \hbar_{\Lambda_K}(d\lambda) \hbar_{\Lambda_K}(d\mu) \\ &= \iint_{\widehat{G} \times \widehat{G}} \overline{\langle \chi, t \rangle} \langle \beta, s \rangle F^P(d\chi, d\beta) \end{aligned}$$

where F^P is the measure on $\widehat{G} \times \widehat{G}$ concentrated on $\Lambda_K \times \Lambda_K$ defined by

$$F^P(\Delta) := \iint_{\Delta \cap (\Lambda_K \times \Lambda_K)} (\widehat{P}_K(\chi), \widehat{P}_K(\beta))_{\mathcal{H}} \hbar_{\Lambda_K}(d\chi) \hbar_{\Lambda_K}(d\beta), \quad \Delta \in \mathcal{B}(\widehat{G} \times \widehat{G}).$$

From relation (12) we find out that the measure γ_λ^P or more precisely its image $F_\lambda^P := \gamma_\lambda^P \circ \ell_\lambda^{-1}$ through the mapping $\ell_\lambda : \widehat{G} \rightarrow \widehat{G}^2$ defined by $\ell_\lambda(\chi) = (\chi, \chi - \lambda)$, $\chi \in \widehat{G}$, is the restriction of the measure F^P to the hyperplane $L_\lambda := \{(\chi, \chi - \lambda) : \chi \in \widehat{G}\}$.

More generally, when X is a PC field, the family $\{\gamma_\lambda : \lambda \in \Lambda_K\}$ is commonly referred to as the *SO-spectral family of X* . The measure γ_λ or more precisely its image $F_\lambda = \gamma_\lambda \circ \ell_\lambda^{-1}$ represents the part of the SO-spectrum of X that sits on the hyperplane $L_\lambda = \{(\chi, \chi - \lambda) : \chi \in \widehat{G}\}$, see relation (10).

Next, we give a sufficient condition for a PC field to be harmonizable.

Theorem 3 *Let X be a G/K -square integrable K -PC field that satisfies the condition [A] of Theorem 1, and let $\{\gamma_\lambda : \lambda \in \Lambda_K\}$ be the SO-spectral family of X . Suppose that there is an \hbar_{Λ_K} -integrable non-negative function ω on Λ_K such that for every $\lambda \in \Lambda_K$*

$$|\gamma_\lambda(\Delta)| \leq \omega(\lambda) \quad \text{for any Borel } \Delta \in \mathcal{B}(\widehat{G}). \quad (13)$$

Then the field X is harmonizable and the SO-spectral measure of X is given by

$$F(\Delta) = \int_{\Lambda_K} F_\lambda(\Delta) \hbar_{\Lambda_K}(d\lambda), \quad \text{for any Borel } \Delta \in \mathcal{B}(\widehat{G} \times \widehat{G}), \quad (14)$$

where $F_\lambda := \gamma_\lambda \circ \ell_\lambda^{-1}$ and $\ell_\lambda(\chi) := (\chi, \chi - \lambda)$, $\chi \in \widehat{G}$.

Notice that condition (13) is satisfied by any G/K -square integrable K -periodic field P such that \widehat{P}_K is \hbar_{Λ_K} -integrable. Here we can take $\omega(\lambda)$ equal to the total variation of the SO-spectral measure γ_λ^P of the field P

$$\omega(\lambda) = \int_{\Lambda_K} \left| \left(\widehat{P}_K(\chi), \widehat{P}_K(\chi - \lambda) \right)_{\mathcal{H}} \right| \hbar_{\Lambda_K}(d\chi).$$

The integrability condition on $\widehat{P}_K : \Lambda_K \rightarrow \mathcal{H}$ is always satisfied when Λ_K is compact that is when G/K is discrete, and in particular when $G = \mathbb{Z}^n$.

Proof. of Theorem 3. Let $\{Z^\lambda : \lambda \in \Lambda_K\}$ be as in Theorem 1. From the proof of Theorem 2 it follows that

$$\gamma_\lambda(\Delta) = \Gamma^{0, -\lambda}(\Delta) = (E(\Delta)Z^0(0), Z^\lambda(0))_{\mathcal{K}}$$

Thanks to definition (7) and Lebesgue dominated convergence theorem it follows that the field $\Lambda_K \ni \lambda \mapsto Z^\lambda(0) \in \mathcal{K}$ is continuous, and hence by assumption (13), $\lambda \mapsto \gamma_\lambda(\Delta)$ is integrable over Λ_K for every Borel Δ of \widehat{G} . For all Borel $D \subseteq \Lambda_K$ and $\Delta \subseteq \widehat{G}$ let us define

$$\tilde{F}(\Delta \times D) := \int_D \gamma_\lambda(\Delta) \hbar_{\Lambda_K}(d\lambda) = \int_D (E(\Delta)Z^0(0), Z^\lambda(0))_{\mathcal{K}} \hbar_{\Lambda_K}(d\lambda).$$

Condition (13) and again Lebesgue dominated convergence theorem entail that the function $\tilde{F}(\Delta \times D)$ is countably additive in Δ and D separately. So to show that the bimeasure \tilde{F} extends to a Borel measure on $\widehat{G} \times \Lambda_K$, it is sufficient to show that its Vitali variation is finite (see [9, 31]), that is

$$\sup \left\{ \sum_{i=1}^n \sum_{j=1}^n |\tilde{F}(\Delta_i \times D_j)| : \Delta_i \cap \Delta_j = \emptyset \text{ and } D_i \cap D_j = \emptyset \text{ for } i \neq j \text{ in } \{1, \dots, n\} \right\} < \infty.$$

Since $\sum_{i=1}^n |\gamma_\lambda(\Delta_j)| \leq \text{Var}(\gamma_\lambda) \leq 4\omega(\lambda)$ and the function ω is \hbar_λ -integrable,

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n |\tilde{F}(\Delta_i \times D_j)| &\leq \sum_{j=1}^n \int_{D_j} \sum_{i=1}^n |\gamma_\lambda(\Delta_i)| \hbar_{\Lambda_K}(d\lambda) \\ &\leq \int_{\bigcup_j D_j} 4\omega(\lambda) \hbar_{\Lambda_K}(d\lambda) \leq \int_{\Lambda_K} 4\omega(\lambda) \hbar_{\Lambda_K}(d\lambda) < \infty. \end{aligned}$$

Hence \tilde{F} is a measure and in particular Fubini and Lebesgue dominated convergence theorems hold for \tilde{F} . Let $\Delta \subseteq \widehat{G}$ be fixed and let $\varphi(\lambda) := \sum_{j=1}^n b_j 1_{D_j}(\lambda)$ be a simple function on Λ_K . Then

$$\int_{\Lambda_K} \varphi(\lambda) \tilde{F}(\Delta, d\lambda) = \sum_{j=1}^n b_j \int_{D_j} \gamma_\lambda(\Delta) \hbar_{\Lambda_K}(d\lambda) = \int_{\Lambda_K} \varphi(\lambda) \gamma_\lambda(\Delta) \hbar_{\Lambda_K}(d\lambda).$$

From condition (13) we deduce that $\int_{\Lambda_K} \varphi(\lambda) \tilde{F}(\Delta, d\lambda) = \int_{\Lambda_K} \varphi(\lambda) \gamma_\lambda(\Delta) \hbar_{\Lambda_K}(d\lambda)$ for any bounded Borel function φ . Consequently, for any simple function ϕ on \widehat{G} and bounded φ on Λ_K

$$\iint_{\widehat{G} \times \Lambda_K} \phi(\chi) \varphi(\lambda) \tilde{F}(d\chi, d\lambda) = \int_{\Lambda_K} \varphi(\lambda) \left[\int_{\widehat{G}} \phi(\chi) \gamma_\lambda(d\chi) \right] \hbar_{\Lambda_K}(d\lambda). \quad (15)$$

If $|\phi|$ is bounded by some finite $c > 0$ then by condition (13), the integral $\int_{\widehat{G}} |\phi(\chi)| \gamma_\lambda(d\chi)$ is bounded by $4c\omega(\lambda)$ which is an \hbar_λ -integrable function of λ . Therefore by Lebesgue dominated convergence theorem, relation (15) holds for any two bounded measurable functions ϕ on \widehat{G} and φ on Λ_K . In particular

$$\begin{aligned} \iint_{\widehat{G} \times \Lambda_K} \overline{\langle \chi, t \rangle} \overline{\langle \lambda, x \rangle} \tilde{F}(d\chi, d\lambda) &= \int_{\Lambda_K} \overline{\langle \lambda, x \rangle} \left[\int_{\widehat{G}} \overline{\langle \chi, t \rangle} \gamma_\lambda(d\chi) \right] \hbar_{\Lambda_K}(d\lambda) \\ &= \int_{\Lambda_K} \overline{\langle \lambda, x \rangle} a_\lambda(t) \hbar_{\Lambda_K}(d\lambda) = \mathbf{B}_X(t; x) = \mathbf{K}_X(t + x, x) \end{aligned} \quad (16)$$

for all $t \in G$ and $x \in G/K$.

Let $\ell : \widehat{G} \times \Lambda_K \rightarrow \widehat{G}^2$ be defined by $\ell(\chi, \lambda) := \ell_\lambda(\chi) = (\chi, \chi - \lambda)$, and let $F = \tilde{F} \circ \ell^{-1}$ be the image of the measure \tilde{F} through the mapping ℓ , that is $F(\Delta) = \tilde{F}\{(\chi, \lambda) : (\chi, \chi - \lambda) \in \Delta\}$. Then F is a Borel measure on \widehat{G}^2 and change of variables formula yields that

$$\iint_{\widehat{G} \times \widehat{G}} \psi(\chi_1, \chi_2) F(d\chi_1, d\chi_2) = \iint_{\widehat{G} \times \Lambda_K} \psi(\chi, \chi - \lambda) \tilde{F}(d\chi, d\lambda) \quad (17)$$

for any bounded Borel function $\psi : \widehat{G} \times \widehat{G} \rightarrow \mathbb{C}$. In particular, in view of relation (16)

$$\begin{aligned} \iint_{\widehat{G} \times \Lambda_K} \overline{\langle \chi, t \rangle} \langle \lambda, s \rangle F(d\chi, d\lambda) &= \iint_{\widehat{G} \times \Lambda_K} \overline{\langle \chi, t \rangle} \langle (\chi - \lambda), s \rangle \tilde{F}(d\chi, d\lambda) \\ &= \mathbf{B}_X(t - s; \iota(s)) = \mathbf{K}_X(t, s). \end{aligned}$$

for all $t, s \in G$ (recall that $\langle \lambda, s \rangle = \langle \lambda, \iota(s) \rangle$ for all $\lambda \in \Lambda_K$ and $s \in G$). Thus the field X is harmonizable and F is its SO-spectral measure. Note that relation (15) holds true if the product $\phi(\chi)\varphi(\lambda)$ is replaced by any bounded measurable function $\psi(\chi, \lambda)$ of two variables. Thanks to such upgraded relation (15) and to relation (17) with $\psi = 1_\Delta$, we get

$$F(\Delta) = \iint_{\widehat{G} \times \Lambda_K} 1_\Delta(\chi, \chi - \lambda) \tilde{F}(d\chi, d\lambda) = \int_{\Lambda_K} \left[\int_{\widehat{G}} 1_\Delta(\chi, \chi - \lambda) \gamma_\lambda(d\chi) \right] \hbar_{\Lambda_K}(d\lambda)$$

So, by the definition of \tilde{F} , we deduce relation (14). ■

Note that if $G = \mathbb{Z}^n$ then condition [A] is satisfied, Λ_K is compact, and condition (13) holds true with $\omega(\lambda) = \text{Var}(\gamma_\lambda) < \infty$. Therefore we generalize the property of harmonizability of the PC sequences proved in [12].

Corollary 1 *Any G/K -square integrable K -PC field over $G = \mathbb{Z}^n$ is harmonizable.*

All the results above simplify significantly if G/K is compact, because then every K -PC field over G is G/K -square integrable and condition [A] in Theorem 1 is always satisfied.

Theorem 4 Suppose that X is a K -PC field and that G/K is compact. Then for every $\lambda \in \Lambda_K$ there is a Borel complex measure γ_λ on \widehat{G} such that

$$a_\lambda(t) = \int_{\widehat{G}} \overline{\langle \chi, t \rangle} \gamma_\lambda(d\chi), \quad t \in G.$$

Moreover the set Λ_K is countable, $\sum_{\lambda \in \Lambda_K} |a_\lambda(t)|^2 < \infty$ and for every $t \in G$

$$\mathbf{B}_X(t; x) \stackrel{L^2}{=} \sum_{\lambda \in \Lambda_K} \overline{\langle \lambda, x \rangle} a_\lambda(t) \quad (18)$$

(the series (18) converges in $L^2(G/K)$ with respect to x). Additionally:

- (i) if $\sum_{\lambda \in \Lambda_K} |a_\lambda(t)| < \infty$ for every $t \in G$, then the series (18) converges also pointwise and uniformly with respect to $x \in G/K$, and for all $t, s \in G$ we have

$$\mathbf{K}_X(t + s, s) = \sum_{\lambda \in \Lambda_K} \overline{\langle \lambda, s \rangle} a_\lambda(t);$$

- (ii) if $\sum_{\lambda \in \Lambda_K} \text{Var}(\gamma_\lambda) < \infty$, then X is harmonizable, and for all $t, s \in G$ we have

$$\mathbf{K}_X(t, s) = \iint_{\widehat{G} \times \widehat{G}} \overline{\langle \chi, t \rangle} \langle \varrho, s \rangle F(d\chi, d\varrho),$$

where the SO -spectral measure F is given by $F(\Delta) = \sum_{\lambda \in \Lambda_K} F_\lambda(\Delta)$, $F_\lambda := \gamma_\lambda \circ \ell_\lambda^{-1}$ and $\ell_\lambda : \widehat{G} \rightarrow \widehat{G} \times \widehat{G}$ is defined as $\ell_\lambda(\chi) := (\chi, \chi - \lambda)$.

Proof. Existence of γ_λ follows from Theorem 2. Since for each $t \in G$ the function $x \mapsto \mathbf{B}_X(t; x)$ is bounded, it is in $L^2(G/K)$ and hence its Fourier transform $\lambda \mapsto a_\lambda(t)$ is in $L^2(\Lambda_K)$. Formula (18) is just the inverse formula for $\mathbf{B}_X(t; \cdot)$. Item (i) follows from the uniqueness of the Fourier transform, while item (ii) from Theorem 3. ■

5 Structure of PC fields

When X is a K -PC field then for every $k \in K$ the mapping $V^k : X(t) \mapsto X(t + k)$, $t \in G$, is well defined and extends linearly to an isometry from $\mathcal{H}_X = \overline{\text{span}} \{X(t) : t \in G\}$ onto itself. The group $\mathcal{V} := \{V^k : k \in K\}$ is a unitary representation of K in \mathcal{H}_X and is called the K -shift of X .

Theorem 5 A continuous field X over G is K -PC if and only if there are a unitary representation $\mathcal{U} = \{U^t : t \in G\}$ of G in \mathcal{H}_X , and a continuous K -periodic field P over G with values in \mathcal{H}_X such that $X(t) = U^t P(t)$, $t \in G$.

Proof. The "if" part is obvious. Prove the other part. Let X be a K -PC field, $\mathcal{V} = \{V^k : k \in K\}$ be the K -shift of X , and E be the spectral resolution of \mathcal{V} . Hence E is a w.c.a.o.s. Borel operator-valued measure defined on \widehat{K} . Since \widehat{K} is isomorphic to \widehat{G}/Λ_K , the measure E can be seen as a measure on \widehat{G}/Λ_K , see [35, Section 2.1.2]. Let ζ be a cross-section for \widehat{G}/Λ_K . For every Borel subset

Δ of \widehat{G} let us define $\tilde{E}(\Delta) := E(\zeta^{-1}(\Delta))$. Then \tilde{E} is a w.c.a.o.s. Borel operator-valued measure on \widehat{G} whose support is contained in a measurable set $\zeta(\widehat{G}/\Lambda_K)$, and whose values are orthogonal projections in \mathcal{H}_X . Since for all $\chi \in \widehat{K}$ and $k \in K$, $\langle \chi, k \rangle = \langle \zeta(\chi), k \rangle$, by change of variable we obtain that

$$V^k = \int_{\widehat{G}} \overline{\langle \chi, k \rangle} \tilde{E}(d\chi), \quad k \in K.$$

Following Gladyshev's idea ([12]) for every $t \in G$ define the operator on \mathcal{H}_X ,

$$U^t := \int_{\widehat{G}} \overline{\langle \chi, t \rangle} \tilde{E}(d\chi), \quad t \in G.$$

Clearly $\mathcal{U} := \{U^t : t \in G\}$ is a group of unitary operators indexed by G . Moreover for every $v \in \mathcal{H}_X$,

$$\|(U^t - I)v\|_{\mathcal{H}}^2 = \int_{\widehat{G}} |\overline{\langle \chi, t \rangle} - 1|^2 \mu_v(dx)$$

where $\mu_v(dx) = \|E(dx)v\|_{\mathcal{H}}^2$ is a finite non-negative measure on \widehat{G} . From Lebesgue dominated convergence theorem we therefore conclude that the unitary operator group \mathcal{U} is continuous, and hence it is a unitary representation of G in \mathcal{H}_X . Note that for $t = k \in K$, we have $U^k = V^k$. Define $P(t) := U^{-t}X(t)$, $t \in G$. Then P is continuous and

$$P(t+k) = U^{-t}U^{-k}X(t+k) = U^{-t}V^{-k}X(t+k) = U^{-t}X(t) = P(t), \quad t \in G, k \in K.$$

So P is a continuous K -periodic field with values \mathcal{H}_X and $X(t) = U^tP(t)$, for every $t \in G$. ■

Theorem 5 gives a good insight on the origin of the measures γ_λ , $\lambda \in \Lambda_K$. Indeed let X be a G/K -square integrable K -PC field, \mathcal{U} and P be as defined in Theorem 5. Then $\|X(t)\|_{\mathcal{H}} = \|P(t)\|_{\mathcal{H}}$ and $(X(t+u), X(u))_{\mathcal{H}} = (U^tP(t+u), P(u))_{\mathcal{H}}$ for all $t, u \in G$. The PC field X being G/K -square integrable, the field $P_K : G/K \rightarrow \mathcal{H}_X$ defined by $P = P_K \circ \iota$ is square integrable as well as the field $x \mapsto U^tP_K(\iota(t) + x)$ defined on G/K , for any $t \in G$. Denoting by $\widehat{P}_K : \Lambda_K \rightarrow \mathcal{H}_X$ the Fourier Plancherel transform of P_K , the Fourier Plancherel transform of $U^tP_K(\iota(t) + \cdot)$ coincides with the function

$$\mu \mapsto \int_{\widehat{G}} \overline{\langle \chi + \mu, t \rangle} \tilde{E}(d\chi) \widehat{P}_K(\mu).$$

where \tilde{E} is the Borel operator-valued measure on \widehat{G} defined in the proof of Theorem 5. Then thanks to Parseval equality, the spectral covariance function of the PC field X verifies

$$\begin{aligned} a_\lambda(t) &= \int_{G/K} \langle \lambda, x \rangle (U^tP_K(\iota(t) + x), P_K(x))_{\mathcal{H}} \hbar_{G/K}(dx) \\ &= \int_{\Lambda_K} \left(\int_{\widehat{G}} \overline{\langle \chi + \mu, t \rangle} \tilde{E}(d\chi) \widehat{P}_K(\mu), \widehat{P}_K(\mu - \lambda) \right)_{\mathcal{H}} \hbar_{\Lambda_K}(d\mu) \\ &= \int_{\widehat{G}} \overline{\langle \rho, t \rangle} \left(\int_{\Lambda_K} \left(\tilde{E}(d\rho - \mu) \widehat{P}_K(\mu), \widehat{P}_K(\mu - \lambda) \right)_{\mathcal{H}} \hbar_{\Lambda_K}(d\mu) \right) \end{aligned}$$

for all $\lambda \in \Lambda_K$ and $t \in G$. The SO-spectral measure of the field X is

$$\gamma_\lambda(\Delta) = \int_{\Lambda_K} \left(\tilde{E}(\Delta - \mu) \widehat{P}_K(\mu), \widehat{P}_K(\mu - \lambda) \right)_{\mathcal{H}} \hbar_{\Lambda_K}(d\mu), \quad \Delta \in \mathcal{B}(\widehat{G}), \lambda \in \Lambda_K.$$

In comparison with expression (12), we see that the spectral resolution \tilde{E} of the unitary operators group \mathcal{U} , in some sense, "spreads" the SO-spectral measure γ_λ^P over \widehat{G} to form γ_λ .

Theorem 5 also suggests a possibility to decompose a PC field into stationary components. If the K -PC field X is G/K -square integrable, we can therefore *formally* write

$$X(t) \approx \int_{\Lambda_K} \overline{\langle \lambda, t \rangle} X^\lambda(t) \hbar_{\Lambda_K}(d\lambda) \quad (19)$$

where $\{X^\lambda(t) := U^t \widehat{P}_K(\lambda) : t \in G\}$, $\lambda \in \Lambda_K$, is a family of jointly stationary fields over G . If in addition \widehat{P}_K is integrable, then integral (19) exists, and we have equality for any t . The integrability condition on \widehat{P}_K being satisfied if Λ_K is compact, that is in particular when $G = \mathbb{Z}^n$, we deduce the following result.

Corollary 2 *Let X be a G/K -square integrable K -PC field over $G = \mathbb{Z}^n$. Then there exists a family $\{X^\lambda : \lambda \in \Lambda_K\}$ of jointly stationary fields over G in \mathcal{H}_X such that*

$$X(t) = \int_{\Lambda_K} e^{i\lambda t'} X^\lambda(t) \hbar_{\Lambda_K}(d\lambda), \quad t \in G,$$

Note that the pair (\mathcal{U}, P) in Theorem 5 is highly non-unique since there are many ways to extend $\mathcal{V} = \{V^k : k \in K\}$ into $\mathcal{U} = \{U^t : t \in G\}$. Consequently the family $\{X^\lambda : \lambda \in \Lambda_K\}$ above is likewise not unique.

If G/K is compact, then every K -PC field is G/K -square integrable, Λ_K is countable, and the integrals above become series. If $G = \mathbb{Z}^n$ and G/K is compact (and hence finite), then Λ_K is finite and Corollary 2 yields the following \mathbb{Z}^n version of Gladyshev's representation of PC sequences included in [12].

Corollary 3 *Suppose that X is a K -PC field over \mathbb{Z}^n and that \mathbb{Z}^n/K is compact. Then Λ_K is finite and there is a finite family $\{X^\lambda : \lambda \in \Lambda_K\}$ of jointly stationary fields over G in \mathcal{H}_X such that for every $t \in G$*

$$X(t) = \sum_{\lambda \in \Lambda_K} e^{i\lambda t'} X^\lambda(t).$$

If Λ_K is not compact then even in the case of a periodic function, its Fourier transform does not have to converge everywhere.

6 Examples

In this section $G = \mathbb{Z}^n \times \mathbb{R}^m$, the Haar measure on \mathbb{Z}^n is the counting measure, the Haar measure on \mathbb{R}^m is $dt/(\sqrt{2\pi})^m$ where dt is the Lebesgue measure on \mathbb{R}^m , $\widehat{\mathbb{Z}^n}$ will be identified with $[0, 2\pi)^n$ with addition *mod* 2π , $\widehat{\mathbb{R}^m}$ will be identified with \mathbb{R}^m , the Haar measures on $[0, 2\pi)^n$ is $dt/(2\pi)^n$, $\widehat{G} = [0, 2\pi)^n \times \mathbb{R}^m$, elements of G and \widehat{G} are row vectors, and the value of a character $\chi \in \widehat{G}$ at $t \in G$ is $\langle \chi, t \rangle = e^{-i\lambda t'}$, where t' is transpose of t . Remembering previous sections, in order to

describe the domain of the SO-spectrum of a K -PC field X over G the only task is to identify G/K and $\widehat{G/K}$ as concrete subsets Q and Λ_K of $G = \mathbb{Z}^n \times \mathbb{R}^m$ and $\widehat{G} = [0, 2\pi)^n \times \mathbb{R}^m$, respectively, in the way that the value of character $\lambda \in \Lambda_K$ at $t \in Q$ is still $\langle \lambda, t \rangle = e^{-i\lambda t'}$. This identification, which is obvious when X is coordinate-wise PC, may be less trivial in the case of more complex K . It may be helpful, and is worth, to note that any closed nontrivial subgroup K of $G = \mathbb{Z}^n \times \mathbb{R}^m$ is isomorphic to $\mathbb{Z}^k \times \mathbb{R}^l$ for some $k, l \in \mathbb{N}$ such that $l \leq m$ and $1 \leq k + l \leq n + m$. This isomorphism, which at least in the case of $G = \mathbb{Z}^n$ or $G = \mathbb{R}^m$ can be found by selecting a proper basis for G (see [14, Theorem 9.11 and A.26]), provides a description and a parametrization of the sets Q and Λ_K . As before $[a]_b$ will denote the remained in integer division of a by b , $b > 0$.

First we briefly revisit a one-parameter case and its slight extension.

Example 1 Suppose that X is a PC process with period $T > 0$. Then $K = \{kT : k \in \mathbb{Z}\}$, $\mathbb{R}/K = [0, T)$ with addition modulo T and $\Lambda_K = \{\frac{2\pi k}{T} : k \in \mathbb{Z}\}$. Moreover for every $\lambda = \frac{2\pi k}{T} \in \Lambda_K$, $a_\lambda(t) := a_k(t) = \frac{1}{T} \int_0^T e^{-is\frac{2\pi k}{T}} \mathbf{K}_X(t+s, s) ds$ and there is a measure γ_k on \mathbb{R} such that $a_k(t) = \int_{\mathbb{R}} e^{itu} \gamma_k(du)$ (Theorem 2). The domain of the SO-spectrum of X is $L = \bigcup_{k \in \mathbb{Z}} L_k$, where $L_k := \{(u, u - \frac{2\pi k}{T}) : u \in \mathbb{R}\}$. The part of the SO-spectrum that sits on L_k is a measure F_k defined as $F_k = \gamma_k \circ \ell_k^{-1}$ where $\ell_k : \mathbb{R} \rightarrow \mathbb{R}^2$, $\ell_k(u) = (u, u - \frac{2\pi k}{T})$. If $\sum_k \text{Var}(\gamma_k) < \infty$ then the process X is harmonizable and $F := \sum_k F_k$ is a measure on \mathbb{R}^2 which satisfies relation (5).

If X is stationary then $K = \mathbb{R}$, $\mathbb{R}/K = \{0\}$, $\Lambda_K = \{0\}$, $a_0(t) = \int_{\{0\}} \mathbf{K}_X(t+s, s) \delta_0(ds) = \mathbf{K}_X(t, 0)$. By Theorem 2 there is a measure γ_0 on \mathbb{R} such that $a_0(t) = \int_{\mathbb{R}} e^{itu} \gamma_0(du)$. Consequently, the SO-spectrum of X is the measure $F = F_0 = \gamma_0 \circ \ell_0^{-1}$, which sits on the diagonal $L_0 = \{(u, u) : u \in \mathbb{R}\}$.

To see the need for the square integrability assumption, let us add one parameter to the above process; that is, let us consider a field $X = \{X(s, t) : (s, t) \in \mathbb{R}^2\}$ such that $\mathbf{K}_X((s, t), (u, v)) = \mathbf{K}_X((s+T, t), (u+T, v))$, $s, t, u, v \in \mathbb{R}$ ($T > 0$ is fixed). Then $K = \{(kT, 0) : k \in \mathbb{Z}\}$, $\mathbb{R}^2/K = [0, T) \times \mathbb{R}$ with addition modulo T on the first coordinate, $\Lambda_K = \{(\frac{2\pi k}{T}, x) : k \in \mathbb{Z}, x \in \mathbb{R}\}$ and

$$a_\lambda(s, t) := a_{k,x}(s, t) = \frac{1}{T\sqrt{2\pi}} \int_0^T \int_{\mathbb{R}} e^{-i(u\frac{2\pi k}{T} + vx)} \mathbf{K}_X((s+u, t+v), (u, v)) du dv$$

for $\lambda = (\frac{2\pi k}{T}, x) \in \Lambda_K$. The square integrability assumption $\int_0^T [\int_{\mathbb{R}} \|X(s, t)\|_{\mathcal{H}}^2 dt] ds < \infty$ assures that the above integral exists. If it does then, in view of Theorem 2, for every $k \in \mathbb{Z}$ and $x \in \mathbb{R}$ there exists a measure $\gamma_{k,x}$ on \mathbb{R}^2 such that $a_{k,x}(s, t) = \int_{\mathbb{R}^2} e^{i(su+tv)} \gamma_{k,x}(du, dv)$. The domain of the SO-spectrum of X is $L = \bigcup_{k \in \mathbb{Z}} \bigcup_{x \in \mathbb{R}} L_{k,x}$, where $L_{k,x}$ is a two-dimensional plane in \mathbb{R}^4 , $L_{k,x} := \{(u, v, u - \frac{2\pi k}{T}, v - x) : u, v \in \mathbb{R}\}$. The "part" of the SO-spectrum that sits on $L_{k,x}$ is a measure $F_{k,x}$ defined as $F_{k,x} := \gamma_{k,x} \circ \ell_{k,x}^{-1}$, where $\ell_{k,x} : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ is defined by $\ell_{k,x}(u, v) := (u, v, u - \frac{2\pi k}{T}, v - x)$. If $\text{Var}(F_{k,x}) \leq \omega(k, x)$ and $\sum_k \int_{\mathbb{R}} \omega(k, x) dx < \infty$, then X is harmonizable and the SO-spectral measure of X is $F = \frac{1}{\sqrt{2\pi}} \sum_k \int_{\mathbb{R}} F_{k,x} dx$ (see Theorem 3). Note that L above is, in fact, the union of countably many three-dimensional hyperplanes D_k in \mathbb{R}^4 , $D_k := \bigcup_{x \in \mathbb{R}} L_{k,x} = \{(u, v, u - \frac{2\pi k}{T}, v - x) : u, v, x \in \mathbb{R}\}$, which are parallel to the "diagonal" D_0 .

If the field $X = \{X(s, t) : (s, t) \in \mathbb{R}^2\}$ is stationary in s , then $\Lambda_K = \{(0, x) : x \in \mathbb{R}\}$, the condition of the square integrability of X means that $\int_{\mathbb{R}} \|X(0, t)\|_{\mathcal{H}}^2 dt < \infty$ and, if the latter is satisfied, the SO-spectrum of X sits on the three-dimensional hyperplane in \mathbb{R}^4 , $L := D_0 = \{(u, v + x, u, x) : u, v, x \in \mathbb{R}\}$. \blacksquare

Next example contains a complete analysis of the SO-spectrum of a weakly PC field.

Example 2 Let T and S be two non-zero integers. Suppose that the field X on \mathbb{Z}^2 is weakly PC with period (T, S) ([21]), that is

$$\mathbf{K}_X((m, n), (u, v)) = \mathbf{K}_X((m + T, n + S), (u + T, v + S)), \quad \text{for all } n, m, u, v \in \mathbb{Z}$$

Here $K = \{k(T, S) : k \in \mathbb{Z}\}$. We assume that at least one of T or S is positive. Let $d := \gcd(T, S)$ be the greatest common positive integer divisor of T and S , so that $(T, S) = d \times (T_1, S_1)$ and $\gcd(T_1, S_1) = 1$. From Bezout's lemma there are integers q, p such that $T_1 q - S_1 p = 1$. Let ϕ be a mapping of \mathbb{Z}^2 onto itself, given by $\phi(m, n) = (m, n)\Phi'$, where $\Phi = \begin{pmatrix} T_1 & p \\ S_1 & q \end{pmatrix}$, and Φ' stands for the transpose matrix of the matrix Φ . Because $\det \Phi = 1$, ϕ is an isomorphism. Since $\phi(dk, 0) = (kT, kS)$ for $k \in \mathbb{Z}$, we have $K = \phi(d\mathbb{Z} \times \{0\})$ and we identify G/K to

$$Q := \phi(\{0, \dots, d-1\} \times \mathbb{Z}) = \{(kT_1 + lp, kS_1 + lq) : k = 0, \dots, d-1, l \in \mathbb{Z}\}.$$

The dual mapping $\psi(s, t) = [(s, t)\Phi^{-1}]_{2\pi} = ([qs - S_1 t]_{2\pi}, [-ps + T_1 t]_{2\pi})$, $s, t \in [0, 2\pi)$, maps $\{\frac{2\pi k}{d} : k = 0, \dots, d-1\} \times [0, 2\pi)$, which is the dual of $\{0, \dots, d-1\} \times \mathbb{Z}$, onto the dual Λ_K of Q . The construction that we use produces a convenient parametrization of Λ_K as the union of d lines : $\Lambda_K = \bigcup_{k=0}^{d-1} \Lambda_k$ where

$$\Lambda_k := \left\{ \left(\left[\frac{2\pi k q}{d} - S_1 t \right]_{2\pi}, \left[\frac{-2\pi k p}{d} + T_1 t \right]_{2\pi} \right) : t \in [0, 2\pi) \right\}.$$

Note that the value of a character $(u, v) = \psi(s, t) \in \Lambda_K$ at $(m, n) = \phi(k, l) \in Q$ is equal to $e^{-i(mu+nv)} = e^{-i(s,t)\Phi^{-1}\Phi(k,l)'} = e^{-i(ks+lt)}$, as required. Assume that X is G/K -square integrable, for example that $\sum_{n=-\infty}^{\infty} \|X(m, n)\|_{\mathcal{H}}^2 < \infty$, for any $m = 1, \dots, T-1$. Then from the previous discussion and the results of Section 4 we deduce the following properties.

- (i) If $(u, v) \in \Lambda_K$, then $u = [\frac{2\pi k q}{d} - S_1 t]_{2\pi}$, $v = [\frac{-2\pi k p}{d} + T_1 t]_{2\pi}$ for some unique $t \in [0, 2\pi)$ and unique $k = 0, \dots, d-1$. Hence the spectral covariance $a_{(u,v)}(m, n) =: a_{k,t}(m, n)$ of X at $(m, n) \in \mathbb{Z}^2$ is equal to

$$a_{k,t}(m, n) = \frac{1}{d} \sum_{j=0}^{d-1} \sum_{l=-\infty}^{\infty} e^{-i(j\frac{2\pi k q}{d} + lt)} \mathbf{K}_X((m + jT_1 + lp, n + jS_1 + lq), (jT_1 + lp, jS_1 + lq)).$$

- (ii) For each $k = 0, \dots, d-1$ and $t \in [0, 2\pi)$ there exists a measure $\gamma_{k,t}$ on $[0, \pi)^2$ such that

$$a_{k,t}(m, n) = \int_0^{2\pi} \int_0^{2\pi} e^{i(mx+ny)} \gamma_{k,t}(dx, dy).$$

- (iii) The SO-spectrum of X sits on the set $L = \bigcup_{k=1}^{d-1} \bigcup_{t \in [0, 2\pi)} L_{k,t}$, where $L_{k,t}$ is a two-dimensional plane in $[0, 2\pi)^4$

$$L_{k,t} := \left\{ \left(x, y, \left[x - \frac{2\pi kq}{d} + S_1 t \right]_{2\pi}, \left[y + \frac{2\pi kp}{d} - T_1 t \right]_{2\pi} \right) : x, y \in [0, 2\pi) \right\}.$$

Note that $D_k = \bigcup_{t \in [0, 2\pi)} L_{k,t}$ is a three-dimensional hyperplane in $[0, 2\pi)^4$, so L is, in fact, the union of d three-dimensional hyperplanes. If $d = \gcd(T, S) = 1$, then $L = D_0 = \left\{ (x, y, [x + St]_{2\pi}, [y - Tt]_{2\pi}) : x, y, t \in [0, 2\pi) \right\}$. In this case the field X is a rotation of the field Y defined by $Y(m, n) := X((m, n)\Phi')$, $m, n \in \mathbb{Z}$, which is stationary in m . Indeed $X(m, n) = Y((m, n)(\Phi')^{-1})$.

- (iv) If there is an integrable function $\omega : [0, 2\pi) \rightarrow [0, \infty)$ such that $\text{Var}(\gamma_{k,t}) \leq \omega(t)$ for all k and t , then X is harmonizable and

$$\mathbf{K}_X((m, n), (j, r)) = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} e^{i(mu + nv - jx - ry)} F(du, dv, dx, dy).$$

The SO-spectral measure F of X is given by $F(\Delta) = \sum_{k=0}^{d-1} \int_0^{2\pi} F_{k,t}(\Delta) dt$, where $F_{k,t}$ is the complex measure on $[0, \pi)^4$ whose support is contained in the plane $L_{k,t}$ and defined by

$$F_{k,t}(\Delta) := \gamma_{k,t} \left\{ (x, y) \in [0, 2\pi)^2 : \left(x, y, \left[x - \frac{2\pi kq}{d} + S_1 t \right]_{2\pi}, \left[y + \frac{2\pi kp}{d} - T_1 t \right]_{2\pi} \right) \in \Delta \right\}.$$

Figure 1 is the graph of the set Λ_K defined previously in the case when $T = 12$ and $S = 9$ ($d = 3$, $p = q = 1$). Then $\Lambda_K = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ consists of three lines, which are shown with different width pattern. If now from each point on the graph we draw the rectangle $[0, 2\pi) \times [0, 2\pi)$ then the resulting three-dimensional body in $[0, 2\pi)^4$ is the domain of the SO-spectrum of the field X . ■

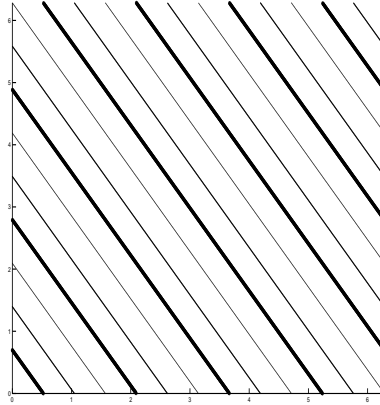


Fig 1: Graph of Λ_K

The last example is a particular example of strongly PC fields over $\mathbb{R} \times \mathbb{Z}^2$. It combines a mixture of continuous and discrete structures.

Example 3 Suppose that X is a field over $G := \mathbb{R} \times \mathbb{Z}^2$ such that $X(t, m, n) = X(t + 4, m, n) = X(t, m + 1, n + 3) = X(t, m + 2, n)$ for all $t \in \mathbb{R}, m, n \in \mathbb{Z}$. Then $K = \{k(4, 0, 0) + j(0, 1, 3) + l(0, 2, 0) : k, l, j \in \mathbb{Z}\}$. In order to describe G/K and Λ_K we consider a change of basis of $G = \mathbb{R} \times \mathbb{Z}^2$ defined by the mapping

$$\phi(t, m, n) := (t, m, n)\Phi', \quad \text{where} \quad \Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$

Then the mapping ϕ is an isomorphism of G onto itself and $K = \phi(P)$, where $P = \{(4k, j, 6l) : k, l, j \in \mathbb{Z}\} = 4\mathbb{Z} \times \mathbb{Z} \times 6\mathbb{Z}$. To see this note that $2(0, 1, 3) - (0, 2, 0) = (0, 0, 6)$, so that K is generated by the 3-tuples $(4, 0, 0)$, $(0, 1, 3)$ and $(0, 0, 6)$, which are respectively equal to $\phi(4, 0, 0)$, $\phi(0, 1, 0)$, and $\phi(0, 0, 6)$. The quotient $G/P = [0, 4) \times \{0\} \times \{0, \dots, 5\}$, so we take $Q := \phi(G/P) = \{(s, 0, l) : s \in [0, 4), l = 0, \dots, 5\}$. The dual of G/P is $\Lambda_P = \frac{2\pi}{T}\mathbb{Z} \times \{0\} \times \{\frac{\pi r}{3} : r = 0, \dots, 5\}$ and hence the dual of G/K can be represented as $\Lambda_K = \psi(\Lambda_P)$, where ψ is the isomorphism of $\hat{G} = \mathbb{R} \times [0, 2\pi)^2$ onto itself defined by $\psi(t, u, v) := (t, u, v)\Phi^{-1} = (t, [u - 3v]_{2\pi}, v)$. Therefore

$$\Lambda_K = \left\{ \left(\frac{2\pi k}{T}, [-\pi r]_{2\pi}, \frac{\pi r}{3} \right) : k \in \mathbb{Z}, r = 0, \dots, 5 \right\},$$

is countable. Note that $[-\pi r]_{2\pi}$ is either π (if r is odd) or 0. For each $\lambda_{k,r} = (\frac{2\pi k}{T}, [-\pi r]_{2\pi}, \frac{\pi r}{3}) \in \Lambda_K$, the corresponding spectral covariance is given by

$$a_{k,r}(t, m, n) = \frac{1}{24} \sum_{l=0}^5 \int_0^4 e^{-i(\frac{2\pi k s}{T} + \frac{\pi r l}{3})} \mathbf{K}_X((t + s, m, n + l), (s, 0, l)) ds,$$

and for each k, r there exists a measure $\gamma_{k,r}$ on $\mathbb{R} \times [0, 2\pi)^2$ such that

$$a_{k,r}(t, m, n) = \int_{\mathbb{R}} \int_0^{2\pi} \int_0^{2\pi} e^{i(ts + mu + nv)} \gamma_{k,r}(ds, du, dv).$$

The SO-spectrum of the field X sits on the union of countably many hyperplanes

$$L_{k,r} := \left\{ \left(s, u, v, s - \frac{2\pi k}{T}, [u + \pi r]_{2\pi}, \left[v - \frac{\pi r}{3} \right]_{2\pi} \right) : s \in \mathbb{R}, u, v \in [0, 2\pi) \right\},$$

$k \in \mathbb{Z}, r = 0, \dots, 5$, of $\mathbb{R} \times [0, 2\pi)^2 \times \mathbb{R} \times [0, 2\pi)^2$. If the sum of total variations of measures $\gamma_{k,r}$ is finite, then X is harmonizable and its SO-spectral measure $F = \sum_{k=-\infty}^{\infty} \sum_{r=0}^5 F_{k,r}$, where $F_{k,r} = \gamma_{k,r} \circ \ell_{k,r}^{-1}$ and $\ell_{k,r}(s, u, v) := (s, u, v, s - \frac{2\pi k}{T}, [u + \pi r]_{2\pi}, [v - \frac{\pi r}{3}]_{2\pi})$.

Note that if we define $Y(t, n, m) := X((t, n, m)\Phi')$, then Y is PC in t with period $T = 4$, stationary in n , and PC in m with period $M = 6$. Since $X(t, n, m) = Y((t, n, m)(\Phi')^{-1})$, one can therefore say that the field X is periodically correlated in direction $(1, 0, 0)$ with period $T = 4$, stationary in direction of $(0, 1, 3)$ and periodically correlated in direction $(0, 0, 1)$ with period $M = 6$. ■

ACKNOWLEDGEMENTS The paper was partially written during the author's stay at Université Rennes 2, Rennes, France, in June 2011.

References

- [1] V.G. Alekseev. On spectral density estimates of Gaussian periodically correlated random fields. *Probab. Math. Statist.* **11** (2) (1991) 157–167.
- [2] T. Bose, M.Q. Chen, K.S. Joo, G.F. Xu. Stability of two-dimensional discrete systems with periodic coefficients. *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing* **45** (7) (July 1998) 839–847.
- [3] T. Bose, R. Thamvichai, T. Radenkovic. Stability of the 2D Fornasini-Marchesini model with periodic coefficients. *IEEE International Conference on Acoustics, Speech, and Signal Processing*, ICASSP'01, Salt Lake City, UT, vol. 3, 7–11 May 2001, 1925–1928.
- [4] W. Chen, G.B. Giannakis, N. Nandhakumar. Spatio-temporal approach for time-varying image motion estimation. *IEEE Trans. Image Process.* **10** (1996) 1448–1461.
- [5] D. Dehay, H.L. Hurd. Representation and estimation for PC and almost PC random processes. In: *Cyclostationarity in Communications and Signal Processing* (Ed. William Gardner), IEEE Press, 1994, 295–329.
- [6] D. Dehay, H.L. Hurd. Spectral estimation for strongly periodically correlated random fields defined on \mathbb{R}^2 . *Mathematical Methods of Statistics* **11** (2002), 135–151.
- [7] Y.P. Dragan, I.N. Yavorskii. The periodic correlation random field as a model for bidimensional ocean waves. *Otbor Peredacha Inform.* **51** (1982) 15–21 (in Russian).
- [8] N. Dinculeanu. *Vector Integration and Stochastic Integration in Banach Spaces*, Wiley, 2000.
- [9] N. Dunford, J. Schwartz. *Linear Operators I and II*, Wiley, 1988 (reedition).
- [10] W.A. Gardner, A. Napolitano, L. Paura. Cyclostationarity: Half a century of research. *Signal Processing* **86**(4) (2006) 639–697.
- [11] P. Gaspar. On Operator periodically correlated random fields. *Operator Theory : Advances and Applications* **153** (2004), 143–156.
- [12] E.G. Gladyshev. Periodically correlated random sequences. *Soviet Math.* **2** (1961), 385–388.
- [13] E.G. Gladyshev. Periodically and almost periodically correlated random processes with continuous time parameter. *Theory Probab. Appl.* **8** (1963), 173–177.
- [14] E. Hewitt, K.A. Ross. *Abstract Harmonic Analysis I and II*, 2nd. Ed., Springer-Verlag, 1979.
- [15] E. Hille. *Functional Analysis and Semi-groups*, A.M.S. Colloquium Pub., Vol. 31, 1948.
- [16] I. Honda. On the spectral representation and related properties of periodically correlated stochastic processes. *Trans. IECE Japan*, E65, **12** (1982), 732–729.
- [17] H.L. Hurd. *An Investigation of Periodically Correlated Stochastic Processes*, Ph.D. Dissertation, Duke University, Durham, North Carolina, 1969.
- [18] H.L. Hurd. Periodically correlated processes with discontinuous correlation functions. *Theory Probab. Appl.* **19** (1974), 804–808.
- [19] H.L. Hurd. Stationarizing properties of random shifts. *SIAM J. Appl. Math.* **26** (1974), 203–211.

- [20] H.L. Hurd. Representation of strongly harmonizable periodically correlated processes and their covariance. *J. Multivariate Anal.* **29** (1989), 53–67.
- [21] H.L. Hurd, G. Kallianpur, J. Farshidi. Correlation and spectral theory for periodically correlated random fields indexed on \mathbb{Z}^2 . *J. Multivariate Anal.* **90** (2004), 359–383.
- [22] H.L. Hurd, G. Kallianpur. Periodically correlated processes and their relationship to $L_1(0, T)$ -valued stationary processes. In: *Nonstationary Stochastic Processes and Their Applications* (Ed. A. G. Miamee), Word Scientific 1991, 256–284.
- [23] H.L. Hurd, A.G. Miamee. *Periodically Correlated Random Sequences; Spectral Theory and Practice*, J. Wiley & Sons, Hoboken, New Jersey, 2007.
- [24] E.T. Kehlet. Cross-sections for quotient maps of locally compact groups. *Math. Scandinavia* **55** (1984), 152–160.
- [25] L.H. Loomis. *An Introduction to Abstract Harmonic Analysis*, Van Nostrand Company, New York, 1953.
- [26] A. Makagon, A.G. Miamee, H. Salehi. Continuous time periodically correlated processes : spectrum and prediction. *Stochastic Process. Appl.* **49** (1994), 277–295.
- [27] A. Makagon. Induced stationary process and structure of locally square integrable periodically correlated processes. *Studia Math.* **136**(1) (1999), 71–86.
- [28] A. Makagon. Characterization of the spectra of periodically correlated processes. *J. Multivariate Anal.* **78**(1) (2001), 1–10.
- [29] A.G. Miamee. PC processes and their stationary dilation. *SIAM J. Appl. Math.* **50** (1990), 1194–1199.
- [30] A.G. Miamee, H. Salehi. On the prediction of periodically correlated stochastic process. In: *Multivariate Analysis V* (Ed. P. R. Krishnaiah), North Holland, Amsterdam, 1980, 167–179.
- [31] M.M. Rao, D. K. Chang. Bimeasures and nonstationary processes. In: *Real and Stochastic Analysis* (Ed. M.M.Rao.), Wiley, 1986, 7–118.
- [32] M.M. Rao. Characterizations of harmonizable fields. *Nonlinear Analysis* **63** (2005), 935–947.
- [33] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups*, Oxford University Press, Oxford, 1968.
- [34] W. Rudin. *Functional Analysis*, McGraw-Hill, New-York, 1973.
- [35] W. Rudin. *Fourier Analysis on Groups*, Wiley, 1990.
- [36] E. Serpedin, F. Panduru, I. Sari, G.B. Giannakis. Bibliography on cyclostationarity. *Signal Processing* **85** (2005) 2233–2303.
- [37] V.S. Varadarajan. *Geometry of Quantum Theory, vol.2*, Springer-Verlag, New York, 1985.